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Compositions of Sasaki Projections

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In an orthomodular lattice (abbreviated OML) L, a Sasaki projection is a mapping $a \to \varphi_x(a) = x \wedge (x^\perp \vee a)$ from L to L, where $x \in L$. We study compositions of finite numbers of Sasaki projections and of the same Sasaki projections composed in inverse order. By using the Baer *-semigroup of all finite compositions of Sasaki projections, we establish a new characterization of kernels of congruences in OMLs and a generalization of the parallelogram law for dimension OMLs. Our results are also related to quantum measurements via Pool's definition of the change of the support of a state after a measurement.

INTRODUCTION

In an orthomodular lattice (abbreviated OML) L, a very important role is played by so-called Sasaki projections, i.e., mappings $a \to \varphi_x(a) = x \wedge (x^\perp \vee a)$ from L to L, where $x \in L$.

In this paper we study compositions of finite numbers of Sasaki projections and of the same Sasaki projections composed in inversed order. As is shown in the Appendix, the set of all finite compositions of Sasaki projections can be endowed with the structure of a Baer *-semigroup which coordinatizes L. Inverting the order of a composition of Sasaki projections corresponds to taking the involution in the Baer *-semigroup. We will establish some relations between Sasaki projections and their "involutions" which generalize the well-known parallelogram law. Our results enable us to characterize kernels of congruence relations in OMLs, so-called orthomodular ideals, as "involution-preserving" ideals. We show that a composition of Sasaki projections and its involution applied to 1 are dimensionally equivalent whenever a dimension equivalence relation can be introduced in an OML. Our results are also related to quantum measurements via Pool's definition of the change of the support of a state after a measurement.

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For notations and basic notions concerning OMLs, the reader is referred to Kalmbach (1983), for foundations of quantum mechanics to Beltrametti and Cassinelli (1981) or Piron (1976), and for Baer *-semigroups and Baer *-rings to Foulis (1960) and Berberian (1972).

1. GENERALITIES ON SASAKI PROJECTIONS

The concept of Sasaki projection was introduced by Sasaki (1951) with the following motivation. One of the most important properties of a Hilbert space H is expressed by the projection theorem:

> If N is a closed subspace of H, then every element a of H can be decomposed as $a = a_1 + a_2$ with $a_1 \in N$ and $a_2 \in N^{\perp}$.

This result can be restated in the language of lattices as follows:

> Let L be the lattice of all closed subspaces of a Hilbert space H and x be an element of L. For every atom a of L there exist two atoms a_1 and a_2 such that $a \leq a_1 \vee a_2$ with $a_1 \leq x$ and $a_2 \leq x^{\perp}$.

Note that if $a \nleq x$ and $a \nleq x^{\perp}$, then a_1 and a_2 are the respective orthogonal projections of a on x.

Now, if a is an element of an OML then we have, for every element x of L, $a \leq \varphi_x(a) \vee \varphi x^{\perp}(a)$ and $\varphi_x(a) \leq x$, $\varphi x^{\perp}(a) \leq x^{\perp}$. If a is an atom of L, then $\varphi_x(a)$ and $\varphi x^{\perp}(a)$ are not necessarily atoms, but if L is an atomic OML satisfying the exchange axiom (Kalmbach, 1983, § 10), then $\varphi_x(a)$ is an atom for every atom a such that $a \nleq x^{\perp}$. Moreover, let P_M be the orthogonal projection on the closed subspace M of a Hilbert space H. For every subspace N, we have $P_M(N) = M \cap (M^{\perp} + N)$ and if N is closed, then $\overline{P_M(N)} = M \cap \overline{(M^{\perp} + N)} = M \wedge (M^{\perp} \vee N)$ in the OML of all closed subspaces of H. Hence, $\overline{P_M(N)} = \varphi_M(N)$ and Sasaki projections are a good generalization of orthogonal projections in a Hilbert space. They possess many other interesting properties and we recall a few of them.

1. Characterization of kernel of congruence relations in OMLs. In an OML, a congruence relation is determined by its kernel, which is called an orthomodular ideal (or a p-ideal). Finch (1966) has proved:

> An ideal I of an OML L is an orthomodular ideal if and only if $a \in I$ and $b \in L$ imply $\varphi_b(a) \in I$.

2. Construction, for every OML L, of a Baer *-semigroup coordinatizing L. Let L be an OML and $S_L = \{\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \mid a_i \in L\}$. Equipped with the

composition of mappings and the two following unary operations,

$$(\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})^* = \varphi_{a_n} \circ \cdots \circ \varphi_{a_1}$$

$$(\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})' = \varphi_d \qquad \text{with} \qquad \mathrm{d} = (\varphi_{a_n} \circ \cdots \circ \varphi_{a_1}(1))^{\perp}$$

S_L is a Baer *-semigroup and L is isomorphic to the OML of all closed projections of S_L. Therefore, every OML can be considered as the OML of all closed projections of a Baer *-semigroup. This fact plays a decisive role in several proofs of Section 3.

More information about Baer *-semigroups can be found in the Appendix.

3. Description of changes of states caused by quantum measurements. In the quantum logic approach to quantum mechanics, the set of all propositions associated with a physical system is supposed to form an orthomodular lattice (a quantum logic) L. According to Pool (1968*a*,*b*), to every proposition p there corresponds a mapping Ω_p of the set $S(L)$ of all states on L into itself, called an operation, and defined in the following way: if s is a state on L such that $s(p) \neq 0$, and a measurement to determine the occurrence or nonoccurrence of p is performed, then $\Omega_p(s)$ is the resulting state in the case that the answer was affirmative.

Recall that an element $a \in L$ is the support of the state s if $a \perp b$ is equivalent to $s(b) = 0$. The support of s, when it exists, is unique and will be denoted by supp(s). If s has a support, then s satisfies the so-called Jauch–Piron property: $s(x) = s(y) = 0$ implies $s(x \vee y) = 0$. Conversely, if L is σ-complete and the state s is completely additive and satisfies the Jauch–Piron property, then s has a support (Kalmbach, 1986, Chapter 2, Theorem 20). Note that if in L every family of pairwise orthogonal elements is finite or denumerable, then each σ-additive state is completely additive. Pool (1986*b*) introduces the following axiom in his definition of an event-state-operation structure:

> Let p be a proposition and s be a state such that $s(p) \neq 0$. If supp(s) and supp($\Omega p(s)$) exist, then supp($\Omega(s)$) $= \varphi_p$(supp(p)).

This axiom is satisfied in the classical case (all the propositions commute) and also in the usual Hilbert space formulation of quantum mechanics. In this formulation, to each physical system $\mathscr{S}$ is attached a Hilbert space H (generally infinite-dimensional, separable, and over the complex field), observables (or physical quantities) of $\mathscr{S}$ are associated with self-adjoint operators of H and states with von Neumann operators (i.e., linear, bounded, self-adjoint, positive, trace class operators of trace one). According to the spectral theorem for self-adjoint operators, every self-adjoint operator A determines a unique vector-valued measure P_A and if A represents an observable $\mathscr{A}$, then the probability that the value of $\mathscr{A}$, when the state of the system is represented by the von Neumann operator D, lies in the Borel set E is $\mathrm{tr}(DP_A(E))$. The state of the system after a measurement is performed to determine if the value of $\mathscr{A}$ belongs to E is represented by $P_A(E)DP_A(E)/\mathrm{tr}(DP_A(E))$ if the state before the measurement was represented by D and in case the answer was affirmative. In this formulation the physical proposition "the value of the observable $\mathscr{A}$ lies in the Borel set E" is in correspondence with the orthogonal projection $P_A(E)$ and the quantum logic associated to the physical system $\mathscr{S}$ is the OML $L = \mathrm{Proj}(H)$, the set of all orthogonal projections of H. Note that a central question in this description is: does every self-adjoint operator represent an observable or, equivalently, is the quantum logic of $\mathscr{S}$ $\mathrm{Proj}(H)$ or only some substructure of $\mathrm{Proj}(H)$? The mappings s_D: $P \in \mathrm{Prof}(H) \to \mathrm{tr}(DP) \in [0, 1]$, where D is a von Neumann operator, are states on the OML $\mathrm{Proj}(H)$ and, by the Gleason theorem, the converse is true if H is separable. In the Baer *-semigroup of all bounded operators of H we have the following equivalences:

$$s_D(P) = 0 \Leftrightarrow \mathrm{tr}(DP) = 0 \Leftrightarrow DP = 0 \Leftrightarrow P = D'P \Leftrightarrow P \leq D' \Leftrightarrow P \perp D''$$

Hence, D'' is the support of the state s_D.

Now, as D is positive, we have $D = X^*X$ for an operator X. Therefore,

$$\begin{aligned} \operatorname{supp} \Omega_P(s_D) &= (PDP)'' = (PX^*XP)'' = ((XP)^*XP)'' \\ &= (X''P)'' = (D''P)'' = \varphi_p(D'') = \varphi_p(\operatorname{supp} s_D) \end{aligned}$$

which justifies the relation postulated by Pool for more general orthomodular lattices.

2. RELATIONS BETWEEN $\varphi_a(b)$ AND $\varphi_b(a)$ IN ORTHOMODULAR LATTICES

Several theorems in the theory of OMLs are called the "parallelogram law for a binary relation R." They mean that $\varphi_a(b)$ and $\varphi_b(a)$ satisfy R and the terminology has its origin in affine geometry: with suitable notations and correspondences the parallelogram law may be interpreted as saying that the four "points" a, $a \wedge b$, b, and $a \vee b$ constitute a parallelogram. In this section, we recall this law for a position P' in general OMLs and the dimensional equivalence relation in dimension OMLs.

Recall that two elements a and b of an orthocomplemented lattice are said to be in position P' if $a^\perp \wedge b = a \wedge b^\perp = 0$. This relation was introduced by J. Dixmier in the lattice of all closed subspaces of a Hilbert space, and comparison of the following two characterizations of this relation is of interest:

1. Two elements a and b of an OML L are in position P' if and only if they are strongly perspective in any subOML containing them (Chevalier, 1983).
2. Two elements a and b of the OML of all projections of a von Neumann algebra are in position P' if they are equivalent (in the sense of Murray and von Neumann) in any subalgebra containing them (Brown, 1958).

The following result shows that $\{\varphi_a(b), \varphi_b(a)\}$ is the general form for a pair of elements of an OML in position P'.

Proposition 1 (Chevalier, 1983). Let a and b be two elements of an OML L. The elements $\varphi_a(b)$ and $\varphi_b(a)$ are in position P', and if x and y are in position P', then $x=\varphi_x(y)$ and $y=\varphi_y(x)$.

Note that, as position P' implies perspectivity and strong perspectivity, these two binary relations also satisfy the parallelogram law.

The pair $\{\varphi_a(b), \varphi_b(a)\}$ also allows us to characterize orthomodular ideals in OMLs:

Proposition 2 (Chevalier, 1986). Let I be an ideal of an OML L. The following conditions are equivalent:

(a) I is an orthomodular ideal.

(b) The binary relation $\varphi_a(b)\in I$ is symmetric [i.e., for any a, $b\in L$, then $\varphi_a(b)\in I$ if and only $\varphi_b(a)\in I$].

If orthomodular lattices have a great interest in quantum logic, they also provide "the lattice theoretic background of the dimension theory of operator algebras" [to use the title of a paper by Loomis (1965)]. Recall the definition of a dimension lattice given in Loomis (1955):

Let L be a complete OML, $\sim$ be an equivalence relation on L. The pair $(L, \sim)$ is called a dimension lattice if the following conditions are satisfied:

(A) If $a\sim 0$, then $a=0$.

(B) (Finite divisibility). If $a_1\perp a_2$ and $b\sim a_1\vee a_2$, then there exists an orthogonal decomposition of b, $b=b_1\vee b_2$, such that $b_1\sim a_1$ and $b_2\sim a_2$.

(C) (Complete additivity). If $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are two families of pairwise orthogonal elements such that $a_i\sim b_i$ for every i, then

$$\bigvee_{i\in I} a_i\sim\bigvee_{i\in I} b_i$$

(D′) If a is perspective to b, then $a\sim b$.

The relation $\sim$ is called a dimensional equivalence relation.

[L. H. Loomis also introduces a condition (D) weaker than (D′).]

The next proposition means that every dimensional equivalence relation satisfies the parallelogram law.

Proposition 3 (Loomis, 1955, Lemma 41). Let a and b be two elements of a dimensional OML $(L, \sim)$. Then $\varphi_a(b) \sim \varphi_b(a)$ holds.

3. RELATIONS BETWEEN $\varphi_{a_1} \circ \cdots \circ \varphi_{a_n}(1)$ AND $\varphi_{a_n} \circ \cdots \circ \varphi_{a_1}(1)$ IN ORTHOMODULAR LATTICES

This section is devoted to the following question: What are the results of the previous section admitting a generalization if $\varphi_{a_1} \circ \cdots \circ \varphi_{a_n}(1)$ and $\varphi_{a_n} \circ \cdots \circ \varphi_{a_1}(1)$ replace $\varphi_a(b)$ and $\varphi_b(a)$? Note that $\varphi_a(b) = \varphi_a \varphi_b(1)$ and $\varphi_b(a) = \varphi_b \varphi_a(1)$.

The following lemma is essential.

Lemma 1. Let $x_1, \ldots, x_n$ be n elements of an OML L. The following equality is satisfied:

$$\varphi_{x_1} \ldots \varphi_{x_n} \varphi_{x_n} \ldots \varphi_{x_1}(1) = \varphi_{x_1} \ldots \varphi_{x_n}(1)$$

Proof. Let S be a Baer *-semigroup coordinatizing L. Then we have in S

$$\begin{aligned} \varphi_{x_1} \ldots \varphi_{x_n} \varphi_{x_n} \ldots \varphi_{x_1}(1) &= (x_1 \ldots x_n \; x_n \ldots x_1)'' \\ &= ((x_n \ldots x_1)^* x_n \ldots x_1)'' \\ &= (x_n \ldots x_1)'' \\ &= \varphi_{x_1} \ldots \varphi_{x_n}(1) \quad \blacksquare \end{aligned}$$

[We used the relation $(x^*x)'' = x''$ satisfied in any Baer *-semigroup; see the Appendix.]

Remark 1. As an immediate consequence of Lemma 1, we obtain that in any OML L, $\varphi_{x_1} \ldots \varphi_{x_n}(1) = 0$ if and only if $\varphi_{x_n} \ldots \varphi_{x_1}(1) = 0$.

This remark enables us to obtain a good generalization of Proposition 2.

Proposition 4. Let I be an ideal of an OML L. The following statements are equivalent:

(a) I is an orthonormal ideal.

(b) For any $a_1, \ldots, a_n \in L$, $\varphi_{a_1} \ldots \varphi_{a_n}(1) \in I$ if and only if $\varphi_{a_n} \ldots \varphi_{a_1}(1) \in I$.

Proof. (a) $\Rightarrow$ (b): Follows easily from Remark 1 applied in L/I.
(b) $\Rightarrow$ (a): Follows from Proposition 2. ■

Remark 2. Let h be a central element of an OML L. As $[0, h]$ is an orthomodular ideal, $\varphi_{a_1} \ldots \varphi_{a_n}(1) \leq h$ if and only if $\varphi_{a_n} \ldots \varphi_{a_1}(1) \leq h$. In particular, if L is complete, then $\varphi_{a_1} \ldots \varphi_{a_n}(1)$ and $\varphi_{a_n} \ldots \varphi_{a_1}(1)$ have the same central cover.

Recall that in a dimension OML $(L, \sim)$, the relation $x \lesssim y$ means that there exists $z \leq y$ such that $x \sim z$.

Lemma 2. Let $(L, \sim)$ be a dimension OML. Then:
(a) For any $a, b \in L$ we have $\varphi_a(b) \lesssim b$
(b) The relation $\sim$ is transitive.

Proof. (a) By Proposition 2, $\varphi_a(b) \sim \varphi_b(a)$ and hence, $\varphi_b(a) \leq b$ implies $\varphi_a(b) \lesssim b$.
(b) Suppose $x \lesssim y$ and $y \lesssim z$. Then there exist $y' \leq y$ and $z' \leq z$ such that $x \sim y'$ and $y \sim z'$. The orthomodular law implies that there exists y'' in L such that $y'' \perp y'$ and $y = y' \vee y''$. From $y \sim z'$ we obtain that there exist z'' and z''' such that $y' \sim z''$, $y'' \sim z'''$, and $z' = z'' \vee z'''$. Transitivity of $\sim$ implies $x \sim z''$, hence $x \lesssim z$. ■

Theorem 1. (Generalization of the parallelogram law.) Let $a_1, \ldots, a_n$ be n elements of a dimension OML $(L, \sim)$. Then we have the relation

$$\varphi_{a_1} \ldots \varphi_{a_n}(1) \sim \varphi_{a_n} \ldots \varphi_{a_1}(1)$$

Proof. By using Lemma 2 n times we infer that

$$\varphi_{a_1} \ldots \varphi_{a_n} \varphi_{a_n} \ldots \varphi_{a_1}(1) \lesssim \varphi_{a_n} \ldots \varphi_{a_1}(1)$$

From this we obtain, by Lemma 1, that $\varphi_{a_1} \ldots \varphi_{a_n}(1) \lesssim \varphi_{a_n} \ldots \varphi_{a_1}(1)$, and the Schröder–Bernstein theorem for dimension OMLs (Loomis, 1955, Lemma 13) entails that $\varphi_{a_1} \ldots \varphi_{a_n}(1) \sim \varphi_{a_n} \ldots \varphi_{a_1}(1)$. ■

Remarks and Examples 3.

1. According to Ramsay (1965), Theorem 8.4, there exists a dimensional equivalence relation on a complete OML L if and only if L satisfies the following covering condition:

$$\text{if } a \text{ covers } a \wedge b, \text{ then } a \vee b \text{ covers } b \tag{1}$$

Note that this condition is equivalent to

$$\text{if } \varphi_a(b) \text{ is an atom, then } \varphi_b(a) \text{ is an atom} \tag{2}$$

Now, consider the condition

$$\text{if } \varphi_{a_1} \ldots \varphi_{a_n}(1) \text{ is an atom, then } \varphi_{a_n} \ldots \varphi_{a_1}(1) \text{ is an atom} \tag{3}$$

Clearly, (3) implies (2). Conversely, consider an OML L satisfying (2). If b is an atom, then, for every $a \in L$, $\varphi_b(a)$ is b or 0. Therefore, by Lemma 1 and (2), $\varphi_a(b)$ is either an atom or 0. Now suppose that $\varphi_{a_1} \ldots \varphi_{a_n}(1)$ is an atom. Applying Lemma 1 and condition (2), we obtain that $\varphi_{a_n} \ldots \varphi_{a_1}(1)$ is an atom and (3) holds. Hence (1)–(3) are equivalent conditions in OMLs.

2. In a dimension OML there exist, in general, several dimensional equivalence relations. For example, let L be the OML of all closed subspaces of a Hilbert space of dimension $\aleph_1$. Define two binary relations $\sim_1$ and $\sim_2$ on L as follows:

$a \sim_1 b \Leftrightarrow a$ and b have the same Hilbert dimension.

$a \sim_2 b \Leftrightarrow a$ and b have the same finite Hilbert dimension or a and b are of infinite Hilbertian dimension.

Clearly, $\sim_1$ and $\sim_2$ are two different dimensional equivalence relations on L. Now, consider a set E of dimensional equivalence relations on an OML L and let $\sim$ be the infimum of E in the lattice of all equivalence relations on L. The equivalence relation $\sim$ satisfies the axioms A, C, and D′ of Loomis and, for $a_1, \ldots, a_n$ in L, $\varphi_{a_1} \ldots \varphi_{a_n}(1) \sim \varphi_{a_n} \ldots \varphi_{a_1}(1)$ holds.

3. Let E be a Euclidean space (or, more generally, a finite-dimensional Hilbert space not necessarily classical). There exists a unique dimensional equivalence relation $\sim$ on the OML L of all subspaces of L defined by $N \sim M$ if and only if M and N have the same algebraic dimension. If f is an endomorphism of E which is a composition of orthogonal projections, $f = P_{N_1} \circ \cdots \circ P_{N_k}$, then we have

$$\begin{aligned} f(E) = P_{N_1} \circ \cdots \circ P_{N_k}(E) &= \varphi_{N_1} \circ \cdots \circ \varphi_{N_k}(E) \\ &\sim \varphi_{N_k} \circ \cdots \circ \varphi_{N_1}(E) = P_{N_k} \circ \cdots \circ P_{N_1}(E) = f^*(E) \end{aligned}$$

where f^* denotes the adjoint of f. In this special case, Theorem 1 means $\operatorname{rank} f = \operatorname{rank} f^*$, a classical result of linear algebra.

4. The proof of Theorem 1 only uses the parallelogram law and the two following properties of $\sim$:

(i) The relation $\sim$ is transitive and satisfies a weak divisibility property: if $a \sim b$, then, for every $c \leq a$, there exists $d \leq b$ such that $c \sim d$.

(ii) There is a theorem of Schröder–Bernstein type for $\sim$:

$$a \lesssim b \text{ and } b \lesssim a \text{ imply } a = b$$

Every binary relation on an OML which satisfies a parallelogram law (i) and (ii) also satisfies a generalized parallelogram law. As an example, one can consider equivalence of projections in a Rickart C^*-algebra (a C^*-algebra whose multiplicative *-semigroup is a Baer *-semigroup) (see Berberian, 1972, § 12, Corollary, and § 13, Theorem 1). Note that, in general, equivalence of projections is not a dimensional equivalence relation on the OML of all projections of a Rickart C^*-algebra (this OML is not necessarily complete and only $\aleph_0$-additivity of equivalence of projections is guaranteed) and therefore Theorem 1 is not available. A similar example is equivalence of projections in a Baer *-ring: every Baer *-ring satisfying the parallelogram law (for equivalence of projections) also satisfies the generalized parallelogram law. There is a little difference between these two examples: as in a Rickart C^*-algebras right and left projections of an element are equivalent (Ara, 1989), a second proof of the generalized parallelogram law for Rickart C^*-algebras will be obtained in Section 4. On the other hand, it is unknown (as far as we know) if a Baer *-ring satisfying the parallelogram law has the property that right and left projections of an element are equivalent.

5. Let L be a locally modular OML. Then, there exists a dimension $\sim$ on L which coincides with strong perspectivity on the set of all elements x such that $[0, x]$ is modular (Ramsay, 1965, Theorem 4.23). Moreover, x is finite for $\sim$ if and only if $[0, x]$ is modular. Therefore, if $\varphi_{a_1} \dots \varphi_{a_n}(1)$ is a finite element of L (this case occurs if one of a_i is finite), then $\varphi_{a_1} \dots \varphi_{a_n}(1)$ and $\varphi_{a_n} \dots \varphi_{a_1}(1)$ are strongly perspective. Since, in a complete modular OML every element is finite for $\sim$, we have proved:

Proposition 5. Let $a_1, \dots, a_n$ be n elements of a complete modular OML. The elements $\varphi_{a_1} \dots \varphi_{a_n}(1)$ and $\varphi_{a_n} \dots \varphi_{a_1}(1)$ are strongly perspective.

Question. Find a common complement of $\varphi_{a_1} \dots \varphi_{a_n}(1)$ and $\varphi_{a_n} \dots \varphi_{a_1}(1)$ for $n \geq 3$.

We have obtained good generalizations for Propositions 2 and 3. The situation is different for Proposition 1 and position P'.

Proposition 6. Let $a_1, \dots, a_n$ be n elements of an OML L ($n \geq 2$). If a_i and a_j commute for any $i \in [2, n-1]$, $j \in [1, n]$, then $\varphi_{a_1} \dots \varphi_{a_n}(1)$ and $\varphi_{a_n} \dots \varphi_{a_1}(1)$ are in position P'.

Proof. Let S be a Baer *-semigroup coordinatizing L. First let us show that in S, $(a_1 \dots a_n)(a_n \dots a_1)'' = (a_1 \dots a_n)(a_n \dots a_1)$. This follows from the equalities

$$\begin{aligned}(a_1 \dots a_n)(a_n \dots a_1)'' &= a_1 \dots a_n a_1 (a_n \dots a_1)'' \qquad [\text{since } a_1 = a_1'' \geq (a_n \dots a_1)''] \\ &= a_1 a_2^2 \dots a_n^2 a_1 (a_n \dots a_1)'' \\ &= (a_1 \dots a_n)(a_n \dots a_1)(a_n \dots a_1)'' \\ &= (a_1 \dots a_n)(a_n \dots a_1)\end{aligned}$$

Now the latter equality implies that

$$\begin{aligned}[(a_1 \dots a_n)''(a_n \dots a_1)'']'' &= [(a_1 \dots a_n)(a_n \dots a_1)'']'' \\ &= [(a_1 \dots a_n)(a_n \dots a_1)]'' \\ &= [(a_n \dots a_1)^*(a_n \dots a_1)]'' \\ &= (a_n \dots a_1)''\end{aligned}$$

In the same way we obtain that

$$[(a_n \dots a_1)''(a_1 \dots a_n)'']'' = (a_1 \dots a_n)''$$

and Proposition 1 implies that $\varphi_{a_1} \dots \varphi_{a_n}(1)$ and $\varphi_{a_n} \dots \varphi_{a_1}(1)$ are in position P'. ■

Remarks 4. (1) If we choose $a_i = 1$ for any $i \in [2, n-1]$, we obtain Proposition 1.

(2) If the assumptions of Proposition 6 hold, then $\varphi_{a_1} \dots \varphi_{a_n}(1) = a_2 \wedge \dots \wedge a_{n-1} \wedge \varphi_{a_1}(a_n)$. In fact,

$$\begin{aligned}(a_n \dots a_1)'' &= [(a_n a_1)'' a_2 \dots a_{n-1}]'' \\ &= [(a_n a_1)'' \wedge a_2 \wedge \dots \wedge a_{n-1}]'' \\ &= (a_n a_1)'' \wedge a_2 \wedge \dots \wedge a_{n-1}\end{aligned}$$

Making use of this result, it is easy to prove that $\varphi_{a_1} \dots \varphi_{a_n}(1)$ and $\varphi_{a_n} \dots \varphi_{a_1}(1)$ are in position P' if one knows that $\varphi_{a_1}(a_n)$ and $\varphi_{a_n}(a_1)$ are in position P'. The proof of Proposition 6 does not utilize the latter result.

(3) It is not possible to weaken the assumptions of Proposition 6. To see this, let us consider Dilworth lattice D_{16} of Figure 1. Put $a_1 = a$, $a_i = b$,

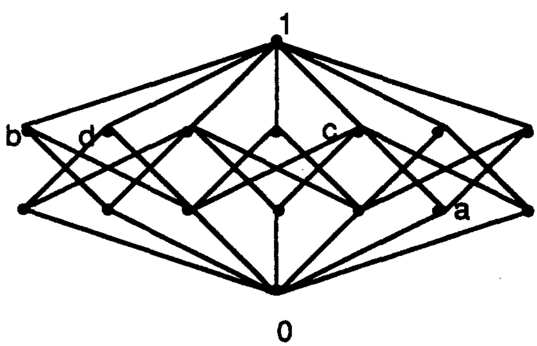

Fig. 1

$a_n = c$, $a_j = 1$ for $j \neq 1$, i, n. Then the only assumption which is not satisfied is the commutativity of a_i and a_n. We have

$$\varphi_a \varphi_b(c) = \varphi_a(b) = a$$

$$\varphi_c \varphi_b(a) = \varphi_c(b) = c \quad \text{and } c \text{ and } a \text{ are not in position } P'$$

4. THE CASE OF PROJECTIONS OF A BAER *-RING

In the OML of all projections of a Baer *-ring there exists a binary relation with properties similar to those of a dimensional equivalence relation on a dimension OML, the equivalence of projections in the sense of Murray and von Neumann. In particular, this relation is an equivalence relation; it satisfies axioms (A) and (B) of Loomis and axiom (C) in the finite case. On the other hand, in the general case, there is no relation between perspectivity and equivalence of projections. Note that two elements p and q of an OML are perspective if and only if $p^{\perp}$ and $q^{\perp}$ are perspective and that the corresponding result for equivalence of projections is false, in general.

This section is devoted to obtaining lattice properties of the pair $\{\varphi_{a_1} \ldots \varphi_{a_n}(1),\ \varphi_{a_n} \ldots \varphi_{a_1}(1)\}$ in an OML L by using the equivalence of projections of a Baer *-ring coordinatizing L.

Let $a_1, \ldots, a_n$ be n elements of an OML L coordinatized by a Baer *-semigroup S. In S, we have

$$\varphi_{a_1} \ldots \varphi_{a_n}(1) = (a_n \ldots a_1)''$$

$$\varphi_{a_n} \ldots \varphi_{a_1}(1) = (a_1 \ldots a_n)'' = (a_n \ldots a_1)^{*''}$$

From this it follows that $\varphi_{a_1} \ldots \varphi_{a_n}(1)$ and $\varphi_{a_n} \ldots \varphi_{a_1}(1)$ are equivalent projections in any Baer *-semigroup in which the axiom $LP \sim RP$ is satisfied (i.e., for any x of S, x'' and $x^{*''}$ are equivalent projections). It is known that this axiom is stronger than the parallelogram law and it holds in every Rickart C^*-algebra (Ara, 1989). To interpret the generalized parallelogram law for equivalence of projections in the language of lattices, we need to consider those Baer *-semigroups in which there is a relation between equivalence of projections and perspectivity.

Recall two conditions concerning *-rings introduced by Maeda and Holland (1976).

(*) For any partial isometry w there exists an invertible element c such that $1 + cw$ is invertible.

(**) For any sequence of orthogonal projections $(e_n)_{n \in N}$ and any projection f such that fe_nf does not depend on n, we have $fe_n = 0$.

These two conditions are satisfied in any unital C^*-algebra.

Proposition 7. Let L be an OML L coordinatized by a Baer *-ring satisfying the parallelogram law (*) and (**). Then, for any $a_1, \ldots, a_n$ in L, there exist elements b_1, b_2, c_1, c_2 in L such that

$$\varphi_{a_1} \ldots \varphi_{a_n}(1) = b_1 \vee b_2, \qquad b_1 \perp b_2$$

$$\varphi_{a_n} \ldots \varphi_{a_1}(1) = c_1 \vee c_2, \qquad c_1 \perp c_2$$

where b_i and c_i are perspective, $i = 1, 2$.

Proof. It is sufficient to apply Lemmas 4 and 6 of Berberian (1984), which give a characterization of equivalence of projections by means of perspectivity in a Baer *-ring satisfying the parallelogram law (*) and (**). ■

Remarks and Examples 5. (1) Every von Neumann algebra, and more generally every AW^*-algebra (a C^*-algebra which is a Baer *-ring), satisfies the assumptions of Proposition 7. Note that Proposition 7 is of interest only if $\varphi_{a_1} \ldots \varphi_{a_n}(1)$ is not a finite projection. If $\varphi_{a_1} \ldots \varphi_{a_n}(1)$ is finite, then Theorem 1 is stronger, since in this case equivalence of projections coincides with perspectivity.

(2) Proposition 7 does not hold in all OMLs. A counterexample can be easily obtained in the Dilworth lattice D_{16}. Indeed, in the notations of Figure 1, we have

$$c = \varphi_c \varphi_d \varphi_a(b^\perp)$$

$$b^\perp = \varphi_b \varphi_a \varphi_d(c)$$

where c and $b^\perp$ are not perspective. In addition, as $b^\perp$ is an atom, we see that Proposition 7 cannot be satisfied. Nevertheless, Proposition 4 suggests that there may exist, in any case, a relation involving perspectivity between $\varphi_{a_1} \ldots \varphi_{a_n}(1)$ and $\varphi_{a_n} \ldots \varphi_{a_1}(1)$.

5. INTERPRETATION IN QUANTUM LOGIC

Notations are those of the final part of Section 1.

Consider a physical system $\mathscr{S}$ and the OML L of all propositions associated with $\mathscr{S}$. If $p_1, \ldots, p_n$ are n propositions, $\varphi_{p_n} \ldots \varphi_{p_1}(1)$ determines the change of the support of a faithful state (i.e., a state with the support 1) after application of the operations $\Omega_{p_1}, \ldots, \Omega_{p_n}$. Theorem 1 says that if we reverse the order of these operations, then the supports of the corresponding states are equivalent with respect to any dimension that can be defined on L. Hence, the change of support of a state is described in the same way in a dimension OML and in a Hilbert space. This result is an argument for the validity of the axiom of J. T. Pool describing the change of support of a state defined on an OML after a measurement.

APPENDIX

A *-semigroup S is a semigroup equipped with an involution $x \to x^*$ satisfying $(xy)^* = y^* x^*$. An element p of S is called a projection if $p = p^2 = p^*$.

A *-semigroup with 0 is said to be a Baer *-semigroup if, for all $a \in S$, the right ideal $\{x \in S \mid ax = 0\}$ is principal and generated by a projection. This projection is unique and denoted a'. The set of all such projections, called closed projections, is denoted by Proj(S). A *-ring in which the multiplicative *-semigroup is a Baer *-semigroup is called a Rickart *-ring and a Baer *-ring if its projection lattice is complete.

As an example of a Baer *-semigroup we can cite $\mathscr{L}(H)$, the semigroup of all bounded operators of a Hilbert space H, and more generally any multiplicative *-semigroup of a von Neumann algebra. Note that in $\mathscr{L}(H)$ we have

$$a' = P_{\ker a}, \qquad a'' = P_{(\ker a)^\perp}, \qquad a^{*\prime} = P_{(\operatorname{Im} a)^\perp}, \qquad a^{*\prime\prime} = P_{\overline{\operatorname{Im} a}}$$

For any Baer *-semigroup S, the binary relation $\leq$ defined by $a \leq b$ if $ab = a$ is an order relation on the set of all projections of S, and equipped with the restriction of the operation $'$ as an orthocomplementation, Proj(S) is an orthomodular lattice.

For the converse, a lemma containing reminiscences from Foulis (1960) and the theory of residuated mappings is useful.

Lemma A1. In every OML L the following properties hold:

(a) $\varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z) \leq a \Leftrightarrow z \leq [\varphi_{x_p} \circ \cdots \circ \varphi_{x_1}(a^\perp)]^\perp$.

(b) $\varphi_{x_1} \circ \cdots \circ \varphi_{x_p}([\varphi_{x_p} \circ \cdots \circ \varphi_{x_1}(a^\perp)]^\perp) \leq a$.

(c) $\varphi_{x_p} \circ \cdots \circ \varphi_{x_1}(a) = \operatorname{Min}\{z \in L \mid \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z) \geq a\}$.

(d) $\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_p} = 0$ is equivalent to $\varphi_{x_1} \circ \cdots \circ \varphi_{x_p} = \varphi_d \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}$ with $d = (\varphi_{a_n} \circ \cdots \circ \varphi_{a_1}(1))^\perp$.

Proof. (a) For $p = 1$, an easy calculation shows that $x_1 \wedge (x_1^\perp \vee z) \leq a$ if and only if $z \leq x_1^\perp \vee (x_1 \wedge a)$. Suppose (a) for an integer p:

$$\varphi_{x_1} \circ \cdots \circ \varphi_{x_p} \circ \varphi_{x_{p+1}}(z) \leq a \Leftrightarrow \varphi_{x_2} \circ \cdots \circ \varphi_{x_{p+1}}(z) \leq [\varphi_{x_1}(a^\perp)]^\perp$$

(by using the case $p = 1$)

$$\Leftrightarrow z \leq [\varphi_{x_{p+1}} \circ \cdots \circ \varphi_{x_2} \circ \varphi_{x_1}(a^\perp)]^\perp$$

(by making use of the hypothesis)

(b) For $p=1$, $\varphi_{x_1}[\varphi_{x_1}(a^\perp)]^\perp = x_1 \wedge a \leq a$. Assume (b) for an integer p:

$$\begin{aligned}
&\varphi_{x_1} \circ \cdots \circ \varphi_{x_{p+1}}([\varphi_{x_{p+1}} \circ \cdots \circ \varphi_{x_1}(a^\perp)]^\perp) \\
&= \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(x_{p+1} \wedge [\varphi_{x_p} \circ \cdots \circ \varphi_{x_1}(a^\perp)]^\perp) \qquad (\text{case } p=1) \\
&\leq \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}([\varphi_{x_p} \circ \cdots \circ \varphi_{x_1}(a^\perp)]^\perp) \leq a
\end{aligned}$$

by making use of the hypothesis and as $\varphi_{x_1} \circ \cdots \circ \varphi_{x_p}$ is increasing

(c) From (a) and (b) we infer

$$[\varphi_{x_p} \circ \cdots \circ \varphi_{x_1}(a^\perp)]^\perp = \mathrm{Max}\{z \in L \mid \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z) \leq a\}$$

which is equivalent to (c)

(d) If $\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_p} = 0$, then, for every z,

$$\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z) = 0$$

and from (a) we infer

$$\varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z) \leq (\varphi_{a_n} \circ \cdots \circ \varphi_{a_1}(1))^\perp = d$$

Hence, $\varphi_d \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z) = \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}(z)$ and $\varphi_d \circ \varphi_{x_1} \circ \cdots \circ \varphi_{x_p} = \varphi_{x_1} \circ \cdots \circ \varphi_{x_p}$ holds. The proof of the converse is similar. ■

Part (c) of Lemma 3 allows us to define an involution $x \to x^*$ on the set $S_L = \{\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} \mid a_i \in L\}$ by

$$(\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})^* = \varphi_{a_n} \circ \cdots \circ \varphi_{a_1}$$

as

$$\varphi_{a_1} \circ \cdots \circ \varphi_{a_n} = \varphi_{b_1} \circ \cdots \circ \varphi_{b_m}$$

implies

$$\varphi_{a_n} \circ \cdots \circ \varphi_{a_1} = \varphi_{b_m} \circ \cdots \circ \varphi_{b_1}$$

Clearly, S_L is a *-semigroups, every Sasaki projection of L is a projection of S_L, and part (d) of the lemma shows that, for an element $\varphi_{a_1} \circ \cdots \circ \varphi_{a_p}$ of S_L, we have

$$\{x \in S_L \mid \varphi_{a_1} \circ \cdots \circ \varphi_{a_p} \circ x = 0\} = \varphi_d S_L$$

with $d = (\varphi_{a_n} \circ \cdots \circ \varphi_{a_1}(1))^\perp$. Hence, S_L is a Baer *-semigroup and, as $(\varphi a^\perp(1))^\perp = a$, the closed projections of S_L are the Sasaki projections of L. Finally, it is easy to prove that the mapping Φ of L into $\mathrm{Proj}(S_L)$ defined by $\Phi(x) = \varphi_x$ satisfies $x \leq y$ if and only if $\Phi(x) \leq \Phi(y)$ [i.e., $\Phi(x) \circ \Phi(y) = \Phi(x)$]. Since $\Phi(x^\perp) = \varphi x^\perp = \varphi'_x$, Φ is an isomorphism of the OML L into $\mathrm{Proj}(S_L)$ which is an orthomodular lattice.

Some elementary properties of Baer *-semigroups used in the proofs of this paper are collected in the following lemma.

Lemma A2. Let a and b be elements of a Baer *-semigroup S and p, q, $p_1, \ldots, p_n$ be closed projections of S.

(a) $aa'=0$, $a'a^*=0$, $ap=0$ is equivalent to $p \leq a'$, $aa''=a$, and $a''a^*=a$.

(b) $pq=qp$ if and only if p and q commute in the OML Proj(S).

(c) If $pq=qp$; then $pq \in \mathrm{Proj}(S)$ and $pq=p \wedge q$.

(d) $(ab)''=(a''b)''$ and $(pq)''=\varphi_q(p)$.

(e) $(a^*a)'=a'$.

(f) $(p_n \ldots p_1)''=\varphi_{p_1} \circ \cdots \circ \varphi_{p_n}(1)$.

Proof. For (a)–(d), see Foulis (1963) or Maeda (1970).

(e) If $ax=0$, then $a^*ax=0$ holds. Conversely, $a^*ax=0$ implies $ax=a^{*\prime}ax$ and, as $a^{*\prime}a=0$, we have $ax=0$ and thus $a'=(a^*a)'$.

(f) By (d), we have

$$\begin{aligned}(p_n \ldots p_1)''&=((p_n \ldots p_2)''p_1)''=\varphi_{p_1}((p_n \ldots p_2)'')\\&=\varphi_{p_1}(((p_n \ldots p_3)''p_2)'')=\varphi_{p_1} \circ \varphi_{p_2}((p_n \ldots p_3)'')\\&=\cdots=\varphi_{p_1} \circ \cdots \circ \varphi_{p_n}(1) \quad \blacksquare\end{aligned}$$

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